Constitutive modeling for sand with emphasis on the evolution of bounding and phase transformation lines

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ABSTRACT: A recently proposed model for drained and undrained behavior of sand under monotonic and cyclic loading conditions is briefly presented. The model is formulated in the framework of classical elastoplasticity, and combines features of: (a) the bounding surface plasticity, (b) the critical state concept, and (c) a hardening evolution law and unloading-reloading rule of the modified Bouc-Wen type. Emphasis is given on the evolution of the bounding, M_s , and phase transformation, M_{pt} , lines. A new set of functions is proposed for the variation of M_s and M_{pt} based on Bolton's relative dilatancy index. It is shown that the new set of M_s and M_{pt} combined with the appropriate calibration according to Bolton (1986), offers a certain degree of flexibility and accuracy that can provide a high level of predictability for both loose and dense states of sand.

1 INTRODUCTION

A recently proposed model for drained and undrained behavior of sand under monotonic and cyclic loading conditions is presented herein. Initially, each constituent of the constitutive formulation is briefly described, such as the elastoplastic matrix, yield function, hardening parameter and plastic flow rule. Emphasis is given on the adoption of the critical state concept in terms of the evolution of the bounding and phase transformation lines. Model predictions are compared with experimental data, as well as, with advanced constitutive model by Dafalias and Manzari, 2004, which incorporates a differently –though widely applied– approach on the evolution of bounding and phase transformation lines.

2 CONSTITUTIVE FORMULATION

2.1 Modified Elastoplasticity

Within the framework of deformation theory of classical elastoplasticity, $d\sigma = \mathbf{E}^{ep} d\mathbf{\epsilon}$, in which \mathbf{E}^{ep} is the elasto-plastic matrix, given by:

$$\mathbf{E}^{ep} = \mathbf{E}^{e} \left[\mathbf{I} - \mathbf{\Phi}_{g} \left(\mathbf{\Phi}_{f}^{T} \mathbf{E}^{e} \mathbf{\Phi}_{g} \right)^{-1} \mathbf{\Phi}_{f}^{T} \right] = \mathbf{E}^{e} \left(\mathbf{I} - \mathbf{B} \right)$$
(1)

in which $\mathbf{\Phi}_{f}$ and $\mathbf{\Phi}_{g}$ account for the failure surface, *f*, and plastic flow rule, respectively:

$$\mathbf{\Phi}_{f} = \frac{\partial f}{\partial \mathbf{\sigma}} \quad , \quad \mathbf{\Phi}_{g} = \frac{\partial g}{\partial \mathbf{\sigma}} \tag{2}$$

Hardening and hysteretic behavior is introduced by inserting the matrices H and η into Eq. (1), as presented by Tasiopoulou and Gerolymos (2012):

$$\mathbf{E}_{h}^{ep} = \mathbf{E}^{e} \left(\mathbf{I} - \mathbf{B} \mathbf{H} \right) \mathbf{\eta} \tag{3}$$

The terms in matrix **H** are functions of the dimensionless hardening parameter ζ , which is of the Bouc-Wen type (Gerolymos and Gazetas, 2005):

$$\mathbf{H} = \boldsymbol{\zeta}^{n} \mathbf{I} \tag{4}$$

in which n is an exponential parameter "controlling" the distance of the current stress state from the failure line.

2.2 Elastic Moduli

The terms in elastic matrix $\boldsymbol{E}^{\boldsymbol{e}}$ include the shear and bulk moduli which are expressed as functions of the mean effective stress \boldsymbol{p} , according to:

$$G = A_o p_{\alpha} \frac{(2.97 - e)^2}{1 + e} \left(\frac{p}{p_{\alpha}}\right)^m, \qquad K = \frac{2(1 + v)}{3(1 - 2v)}G$$
(5)

in which, A_o is a dimensionless material parameter, v is the Poisson's ratio, p_a is the atmospheric pressure, e is the current void ratio and m is a dimensionless parameter determining the rate of variation of G and K with p.

2.3 Yield function

The model incorporates a failure envelope of the Drucker-Prager type, representing the bounding surface:

$$f = \left[(\mathbf{s} - \mathbf{r}_{max} p) : (\mathbf{s} - \mathbf{r}_{max} p) \right]^{1/2} - \left(\sqrt{\frac{2}{3}} M_{s,\vartheta} - n_{max} \right) p = 0$$
 (6)

where $M_{s,\vartheta}$ is the ultimate strength line in q-p space and it is dependent on the Lode angle, according to $M_{s,\vartheta} = \left(\frac{M_c + M_e}{2} - M_{ss}\right) cos^2 (3\vartheta) + \left(\frac{M_c - M_e}{2}\right) cos(3\vartheta) + M_{ss}$,

where M_c is the bounding surface (and ultimate

strength) in compression, $M_{\rm e}$ in extension and $M_{\rm s}$

in simple shear, while $cos(3\vartheta) = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$.

The yield function takes into account both the initial K_o and reversal loading conditions through the stress ratio tensor \mathbf{r}_{max} and stress ratio value n_{max} . The stress ratio tensor \mathbf{r}_{max} is defined as:

$$\mathbf{r}_{max} = \frac{\mathbf{s}_{max}}{p_{max}} = \frac{\mathbf{\sigma}_{max} - p_{max}\mathbf{I}}{p_{max}}$$
(7)

where σ_{max} and p_{max} are equal to the initial stress values at the beginning of loading. In the following, they obtain the stress values at the pivot points once reversal of loading occurs. It is evident that the modified deviatoric stress $q_m = [(\mathbf{s} - \mathbf{r}_{max} p): (\mathbf{s} - \mathbf{r}_{max} p)]^{1/2}$ always obtains zero value at the beginning of loading, for all K_o initial conditions, and at each load reversal. The stress ratio value n_{max} is defined as the inner product of two tensors, such as $n_{max} = \mathbf{n}: \mathbf{r}_{max}$, where **n** is a normalized stress ratio tensor showing the loading direction and it is equal to the derivative of the q_m with respect to $\boldsymbol{\sigma}$; thus, normal to f:

$$\mathbf{n} = \frac{\mathbf{s} - \mathbf{r}_{max} p}{\left[(\mathbf{s} - \mathbf{r}_{max} p) : (\mathbf{s} - \mathbf{r}_{max} p) \right]^{1/2}}$$
(8)

The properties of the tensor **n** are: trn=0 and $trn^2 = n : n = 1$. In either case, n_{max} , incorporates the effects of the deviatoric stress ratio due to initial and pivot-point stress conditions (r_{max}) on the **current** loading direction/path (through current tensor **n**). Therefore, n_{max} , compensates for the return of the deviatoric stress q_m to the hydrostatic axis at each load reversal. The yield surface can be rewritten as:

$$\mathbf{s}:\mathbf{n}-\sqrt{\frac{2}{3}}M_{s,\vartheta}p=0\tag{9}$$

2.4 Hardening Parameter, ζ

Following Eqn. (9), the hardening parameter, ζ , is defined as:

$$\zeta = \frac{\mathbf{s} \cdot \mathbf{n}}{\sqrt{\frac{2}{3}} M_{s,\vartheta} p} \tag{10}$$

The hardening parameter, ζ , is bounded, strictly obtaining values within the range [0 1]. In particular, ζ obtains zero value at the beginning of the loading and at reversal points, which leads to zero values of the hardening matrix **H** according to Eqn. (4). Consequently, the elastoplastic matrix, \mathbf{E}_{h}^{ep} becomes equal to the elastic matrix, \mathbf{E}_{h}^{ep} (3). The gradient to the yield surface [Eqn. (6)] is obtained as:

$$\mathbf{\Phi}_{f} = \frac{\partial f}{\partial \mathbf{\sigma}} = \mathbf{n} - \frac{1}{3} \zeta \sqrt{\frac{2}{3}} M_{s,\vartheta} \mathbf{I}$$
(11)

2.5 Plastic Flow Rule

The stress-dilatancy relationship, adopted by the model, is based on Rowe's dilatancy theory (Rowe 1962). The ratio of the plastic volumetric strain increment, $d\varepsilon_p^{\ \rho}$, over the plastic deviatoric strain increment, $d\varepsilon_q^{\ \rho}$ depends on the distance of the current stress ratio, q/p in conventional p-q space from the phase transformation line, M_{pt} , as follows:

$$\frac{d\varepsilon_p^{\ p}}{\left|d\varepsilon_q^{\ p}\right|} = \left(M_{pt} - \frac{q}{p}\right) \tag{12}$$

The dilatancy in 3D formulation remains a scalar quantity, calculated by:

$$d = \left(\sqrt{\frac{2}{3}}M_{pt,\vartheta}\mathbf{n} - \frac{\mathbf{s}}{p}\right) : \mathbf{n} = \left(\sqrt{\frac{2}{3}}M_{pt,\vartheta} - \frac{\mathbf{s}}{p} : \mathbf{n}\right)$$
(13)

The gradient to the potential function g, considering non-associative plasticity, is given by:

$$\mathbf{\Phi}_{g} = \frac{\partial g}{\partial \mathbf{\sigma}} = \mathbf{n} + \frac{1}{3} d\mathbf{l}$$
(14)

3 EVOLUTION OF BOUNDING & PHASE TRANSFORMATION LINES

3.1 Adopting Critical State Concept

The essence of the critical state concept is that no change in volume occurs when the current stress state reaches the critical state, while the shear deformation continuously increases. In order to achieve this kind of performance upon critical state, both the phase transformation line, M_{pt} and the ultimate strength line (or else bounding surface), M_s , should evolve in p-q space converging to the critical state line, M_{cs} and produce zero plastic volumetric change when $M_{pt} = M_s = M_{cs}$. Several suggestions have been made in literature for the variation of M_s and M_{pt} , based on a suitable current state material parameter relative to the critical state (Wood et al., 1994; Manzari and Dafalias, 1997); one of the most recent and efficient, being proposed by Dafalias and Manzari, 2004 and used also by Taiebat and Dafalias, 2007:

$$M_{s} = M_{cs} e^{-n_{s}\Psi} \quad and \quad M_{pt} = M_{cs} e^{n_{pt}\Psi} \tag{15}$$

where $\Psi = e - e_c$, e_c is the void ratio given by the critical state line in e-p space and n_s , n_{pt} appropriate constants. The effectiveness of Eqns. (15) relies on two satisfied requirements: i) when $e = e_c$ then $M_{pt} = M_s = M_{cs}$ and ii) for denser sands where $e < e_c$, $M_{pt} < M_{cs} < M_s$ and for looser sands where $e > e_c$, then $M_{pt} > M_{cs} > M_s$. Although the concept of Eqns. (15) is flawless, some challenges can appear in calibration process. For example, the values of constants n_s, n_{pt} should simultaneously account for both the accurate prediction of peak strength and dilatancy for all

states of sand which is interlinked with how rapidly the critical state is reached. Apparently, this is a challenging scheme with limited versatility which can lead to a stiffer and more dilative prediction of both dense and –especially– loose sand behaviour compared to experiments, as shown in Figures 1 and 2.



Figure 1. Comparison between experimental data from drained triaxial tests on Toyoura sand and model predictions.

After meticulous observation of experiments and careful consideration of the above, the evolution of the ultimate strength line (bounding surface) was chosen herein, as a function of the accumulation of deviatoric strain increments (chosen hardening parameter):

$$M_{c} = M_{cs} + \left[M_{sp} + \left(M_{so} - M_{sp}\right)e^{-C\sum\left|d\epsilon_{q}\right|} - M_{cs}\right]e^{-C\sum\left|d\epsilon_{q}\right|} \quad (16)$$

where M_{so} is an initial value of the ultimate strength, and M_{sp} is a maximum value that can be potentially reached depending on the model parameter *c*, providing a more flexible shape that can accommodate the behavior of both dense and loose sands. The phase transformation line evolves in the same context, according to the following expression:

$$M_{pt} = M_{cs} + \left(M_{pto} - M_{cs}\right)e^{-0.5c\sum\left|d\varepsilon_q\right|}$$
(17)

in which M_{pto} is the initial value of M_{pt} .

3.2 Calibration based on relative dilatancy index by Bolton 1986

Remaining goal of the calibration is to relate the above mentioned model parameters to physically meaningful parameters in the framework of critical state theory for sands, such as relative density, D_r , and mean effective pressure, p. This goal can be achieved by using Bolton's relative dilatancy index:

$$I_r = D_r \left(Q - \ln(p) \right) - R \tag{18}$$

where D_r is the current relative density of the sand, pis the current mean effective stress, and Q, R are constants which can obtain values close to 10 and 1, respectively. Critical state occurs when $I_r = 0$, while $I_r > 0$ indicates denser states of sands and $I_r < 0$ accounts for looser contractive states. For triaxial strain the maximum friction angle is obtained as: $\varphi_{max} - \varphi_{cs} = 3I_r$. Thus, the real obtained value of M_{speak} should be equal to $M_{speak} = \frac{6\sin(\varphi_{max})}{3-\sin(\varphi_{max})}$. The bounding surface obtains its maximum value ($M_s = M_{speak}$) when $\frac{\partial M_s}{\partial (\sum |d\epsilon_q|)} = 0 \Leftrightarrow \sum |d\epsilon_q|_{peak} = \frac{1}{c} \ln \left(\frac{M_{cs} - M_{sp}}{2(M_{so} - M_{sp})} \right)$,

where $M_{so} = \frac{6\sin(0.8\varphi_{cs} + 5I_{ro})}{3 - \sin(0.8\varphi_{cs} + 5I_{ro})}$, with I_{ro} being the initial

value of the relative dilatancy index, I_r . Thus, M_{sp} can be obtained as a function of M_{speak} : $M_{sp} = 2M_{speak} - M_{cs}$ (10)

$$+0.5\sqrt{\left(2M_{cs}-4M_{speak}\right)^{2}-16M_{so}\left(M_{speak}-M_{cs}\right)+4M_{cs}^{2}}$$
(19)

Bolton (1986) also suggested after observation of numerous drained tests on sands in the range of $0 < I_{ro} < 4$ (dilative sands) that:

$$\left[-\frac{d\varepsilon_{p}}{d\varepsilon_{1}}\right]_{\max}=0.3I_{p}$$

(20)

when the peak strength value, M_{speak} , is obtained; a deduction that can contribute to the calibration of



Figure 2. Comparison between experimental data from undrained triaxial tests on Toyoura sand and model predictions.

the plastic flow rule and specifically the phase transformation line. Applying the flow rule of Eqn. (12) for triaxial tests, we obtain:

$$\left[\frac{d\varepsilon_{p}^{p}}{d\varepsilon_{q}^{p}}\right]_{\max, tx} = M_{pt, peak} - M_{speak} = \frac{-3\left[-\frac{d\varepsilon_{p}}{d\varepsilon_{1}}\right]_{\max}}{3 + \left[-\frac{d\varepsilon_{p}}{d\varepsilon_{1}}\right]_{\max}} = \frac{-3(AI_{r})}{3 + (AI_{r})} \qquad \text{as-}$$

suming that $\left[\frac{d\varepsilon_{\rho}^{p}}{d\varepsilon_{q}^{p}}\right]_{max} \simeq$

$$\left[\frac{d\varepsilon_{p}}{d\varepsilon_{q}}\right]_{\max}$$
 when maximum

strength (failure) has occurred. It is evident from Eq. (20) that A=0.3 for dilative sands. In order to satisfy the above requirement at peak strength $(M_{pt,peak} = M_{speak} - \frac{3(AI_r)}{3 + (AI_r)}), M_{pto}$ was chosen to be expressed as $M_{pto} = M_s \zeta^n - \frac{3(AI_r)}{3 - 3(AI_r)}$. Constant evolution during loading will lead to $M_{pt} \simeq M_{pto} = M_{speak} - \frac{3(AI_r)}{3 - 3(AI_r)}$

at peak strength, where $\zeta = 1$. Parameter A remains to be determined for loose sands $(I_r < 0)$ taking account the fact that the maximum value of the ratio $-\left[\frac{d\varepsilon_p^p}{d\varepsilon_q^p}\right]_{max}$ is obtained at the beginning of the loading and gradually tends to zero as critical state is reached. Figure 3 concentrates experimental values of the ratio $\left[-\frac{d\varepsilon_p}{d\varepsilon_1}\right]_{max}$ as a function of the initial value

of I_r (I_{ro}) for various drained triaxial tests on loose sands. It can be safely assumed that the ratio obtains a constant value of -0.55 independently of I_{ro} . Therefore, the parameter A, for loose sands, can be deter-



Figure 3. Data for the initial ratio of volume strain increment over axial strain increment obtained from drained triaxial tests on different loose sands by Been and Jefferies, 1985 versus the initial value of Bolton's relative dilatancy index, I_{ro} .

3.3 Model Predictions versus Experiments

Figures 1 and 2 illustrate the capacity of the presented model to predict experimental results of drained and undrained monotonic triaxial tests. Furthermore, the model predictions (Tasiopoulou & Gerolymos, 2013) are compared with those of an advanced model by Taiebat & Dafalias (2007), using Eqns. (15). The values of parameters used for the current model are shown in Table 1. Relative density, D_r , was calculated considering e_{max} =0.977 and e_{min} =0.597 for Toyoura sand, according to Verdugo and Ishihara (1996). The flexibility offered by the proposed set of M_s and M_{pt} (Eqns. 16-17) provides better agreement with the experiments especially for loose sands. In case of denser sands, the level of predictability can be considered equal for both models.

Figure 4 offers an insight on how the new presented constitutive formulation would cooperate with the two different approaches of M_s and M_{pt} : i) Eqns. (15) suggested by Dafalias and Manzari (2004) and (ii) Eqns. (16-17) presented herein. The values of parameters of Eqns. (15) along with the required critical state line in e-p space were obtained by Taiebat and Dafalias (2007). The differences are located mostly to the initial values of M_s which in case of the new Eqns. (16-17) are lower than M_{cs} for both the looser and denser sand. According to these new equations, the stress ratio q/p reaches the bounding line early during loading and then it follows the shape of the bounding line up to the critical state. On the other hand, according to the set of functions of Eqns. (15) suggested by Dafalias and Manzari (2007), the stress ratio meets the bounding line later during loading at a higher current value of $M_{\rm s}$ and then follows the bounding line up to critical state in a much slower rate.

4 CONCLUSIONS

A new constitutive model for sands is presented herein, and emphasis is given on the evolution of the bounding, M_s , and phase transformation, M_{pt} , lines. A new set of functions is proposed for the variation of M_s and M_{pt} based on Bolton's relative dilatancy

Table 1. Model parameters and values.

	Parameter	Values
Elasticity	A _o	140
	v	0.15
	т	0.6
Critical State	$\varphi_{cs}(^{\circ})$	31
	M _{cs}	1.244
	Q	9.1
	R	0.77
Hardening	С	10
	n	0.3

index. This currently presented formulation along with the new set of functions M_s and M_{pt} are compared with both experimental results and the advanced constitutive model by Dafalias & Manzari (2004) and Taiebat & Dafalias (2007) using Eqns. (15). It is shown that the new set of M_s and M_{pt} combined with the appropriate calibration according to Bolton (1986), offers a certain degree of flexibility and accuracy that can provide a high level of predictability for both loose and dense states of sand.

5 ACKNOWLEDGEMENT

The financial support for this paper has been provided by the research project "FORENSEIS" implemented under the "ARISTEIA" Action of the "OP-ERATIONAL PROGRAMME EDUCATION AND LIFELONG LEARNING".



Figure 4. Evolution of bounding line M_s , phase transformation line, M_{pt} and stress ratio q/p versus axial strain during drained triaxial simulations in case of a loose and dense sand using the new Eqns. (16-17) for a) and c) and Eqns. (15) by Dafalias and Manzari (2004) for b) and d).

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